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1971 J. Phys. A: Gen. Phys. 4 695

(http://iopscience.iop.org/0022-3689/4/5/012)

J. Phys. A: Gen. Phys., 1971, Vol. 4. Printed in Great Britain

Motions in a Bose condensate
II. The oscillations of the rectilinear and large circular vortex lines

J. GRANT

School of Mathematics, University of Newcastle upon Tyne, Newcastle upon Tyne, NE1 7RU, England

MS. received 23rd April 1971

Abstract. The linearized perturbation equations for the rectilinear and large circular vortex lines in a Bose condensate are studied. An asymptotic matching procedure is given whereby the frequency ω for the (bound state) oscillations of the rectilinear vortex, for wavelengths large compared with the healing length a, is found to be given by

$$\omega = \frac{\kappa k^2}{4\pi}\left\{\ln\left(\frac{2}{ka}\right) - 0{\cdot}692\right\}$$

to the leading order, where κ is the quantum of circulation and $2\pi/k$ is the wavelength. A similar asymptotic procedure is employed to investigate the behaviour of the large ring vortex when its circular axis is slightly perturbed. It is found that the vortex, whose radius c is large compared with a, is stable for all small displacements (and executes simple harmonic vibrations). For the state of oscillation in which there are p waves around the circumference of the ring the frequency is given by the relation

$$\omega^2 = \left(\frac{\kappa}{4\pi c^2}\right)^2 \left[p^2\left\{\ln\left(\frac{8c}{a}\right) - 2f(p) - 0{\cdot}115\right\} + \tfrac{1}{2}f(p)\right]\left[(p^2-1)\left\{\ln\left(\frac{8c}{a}\right) - 2f(p) - 0{\cdot}115\right\} - \tfrac{1}{2}(f(p)-1)\right]$$

to the leading order in $1/c$, where

$$f(p) = 1 + \tfrac{1}{3} + \ldots + \frac{1}{2p-1} \qquad f(0) = 0.$$

In the case of the rectilinear vortex, the eigenvalue spectrum for arbitrary k is studied numerically.

1. Introduction

In a recent paper Roberts and Grant (1971 to be referred to as I) have examined the structure of the large circular vortex in liquid helium, using as a model the Bose condensate which is characterized by a wavefunction Ψ whose normalization yields the total number of particles. This one-particle distribution wavefunction obeys the equation

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2M}\nabla^2\Psi + V_0\Psi|\Psi|^2 \tag{1.1}$$

where M is the particle mass. A short-range repulsive potential V_0 of delta-function type, which was proposed by Ginzburg and Pitaevskii (1958), is incorporated. The total number of particles N is given by

$$N = \int_v |\Psi|^2\,\mathrm{d}V \tag{1.2}$$

and

$$j = \frac{\hbar}{2\mathrm{i}} \int_v (\Psi^* \nabla \Psi - \Psi \nabla \Psi^*) \, \mathrm{d}V \tag{1.3}$$

is the number current density (see Gross 1961 and 1963).

In this paper, which is a sequel to I, the time-dependent oscillations of the rectilinear and large circular quantum vortex lines in the theory are examined by considering small perturbations to the potential and density when the basic equations (1.1) to (1.3) are reduced to fluid mechanical form.

The vibrations of a classical rectilinear vortex were studied by Thomson (1880) who obtained expressions for the frequency of oscillation ω of vortices with various core structures, in the long-wavelength limit. For a hollow irrotational vortex he obtained the result

$$\omega = \frac{\kappa k^2}{4\pi} \left\{ \ln \left(\frac{2}{ka} \right) - \gamma \right\} \tag{1.4}$$

where $2\pi/k$ is the wavelength of the vibration, a is the core radius, κ is the circulation and γ is Euler's constant ($\simeq 0{\cdot}577$), while for a solid-core vortex with solid-body rotation he found

$$\omega = \frac{\kappa k^2}{4\pi} \left\{ \ln \left(\frac{2}{ka} \right) - \gamma + \tfrac{1}{4} \right\}. \tag{1.5}$$

In the Bose condensate theory the oscillations of the quantum rectilinear vortex were first discussed by Pitaevskii (1961). He showed that for $ka \ll 1$ the vibrations are similar to those of an ordinary classical liquid and have the dispersion relation

$$\omega = \frac{\kappa k^2}{4\pi} \ln \left(\frac{1}{ka} \right). \tag{1.6}$$

All of (1.4) to (1.6) are the same to the leading (logarithmic) order as $ka \to 0$, and this shows, physically, that waves of such a large wavelength do not distinguish different core structures. In the next order, however, the core structure makes itself felt, as a comparison of (1.4) and (1.5) will indicate, and one of the objectives of this paper is to refine Pitaevskii's result (1.6) to include this effect. This is achieved by a formal asymptotic analysis. It is found that

$$\omega = \frac{\kappa k^2}{4\pi} \left\{ \ln \left(\frac{2}{ka} \right) - \gamma + \epsilon_c - \tfrac{1}{2} \right\} \tag{1.7}$$

where ϵ_c is the energy per unit length of the 'core' of the vortex in units of $\rho_\infty(\kappa/2\pi)^2$. Here ρ_∞ is the fluid density at infinity and if $\rho_\infty R_0^2$ denotes the radial density function then ϵ_c ($\simeq 0{\cdot}385$) is given by (cf. I equation (4.12))

$$\epsilon_c = \int_0^\infty r \left(\frac{\mathrm{d}R_0}{\mathrm{d}r} \right)^2 \mathrm{d}r + \int\!\!\!\!\!\!- _0^\infty \frac{R_0^2}{r} \, \mathrm{d}r + \tfrac{1}{2} \int_0^\infty r(1 - R_0^2)^2 \, \mathrm{d}r. \tag{1.8}$$

In § 4 a parallel asymptotic analysis is developed to treat the interesting problem of the subsequent behaviour of a large ring-vortex when its circular axis is slightly perturbed. It is found that the vortex ring, whose radius c is large compared with a, is stable for all small displacements. In the state of oscillation in which there are p

waves around the circumference of the ring the frequency is given by

$$\omega^2 = \left(\frac{\kappa}{4\pi c^2}\right)^2 \left[p^2\left\{\ln\left(\frac{8c}{a}\right) - 2f(p) - \tfrac{1}{2} + \epsilon_c\right\} + \tfrac{3}{2}f(p)\right]\left[(p^2-1)\left\{\ln\left(\frac{8c}{a}\right) - 2f(p) - \tfrac{1}{2} + \epsilon_c\right\} - \tfrac{3}{2}\{f(p)-1\}\right] \tag{1.9}$$

to the leading order in c^{-1} where

$$f(p) = \sum_{n=1}^{p} \frac{1}{2n-1} \qquad f(0) = 0.$$

This result may be compared with that obtained by J. J. Thomson who, in his Adam's Prize Essay of 1882 (Thomson 1883), conducted a kinematical investigation of a large classical hollow core vortex ring which was disturbed in the same way. His analysis yielded

$$\omega^2 = \left(\frac{\kappa}{4\pi c^2}\right)^2 \left[p^2\left\{\ln\left(\frac{8c}{a}\right) - 2f(p) - \tfrac{1}{2}\right\} + \tfrac{3}{2}f(p)\right]\left[(p^2-1)\left\{\ln\left(\frac{8c}{a}\right) - 2f(p) - \tfrac{1}{2}\right\} - \tfrac{3}{2}\{f(p)-1\}\right] \tag{1.10}$$

and again it is apparent that the vibrations of the quantum and classical ring vortices are very similar; the frequency being influenced only in the second order by the core structure.

2. The general perturbation equations

Following the theory of (I) the condensate equations (1.1) to (1.3) have been reduced to the following fluid mechanical form:

$$\frac{\partial \rho}{\partial t} = \nabla . (\rho \nabla \phi) \tag{2.1}$$

$$2\frac{\partial \phi}{\partial t} = (\nabla\phi)^2 + \rho - 1 - \frac{\nabla^2 \rho^{1/2}}{\rho^{1/2}} \tag{2.2}$$

where ρ is the fluid density and ϕ is the potential so that the velocity $\boldsymbol{u} = -\nabla\phi$. The unit of circulation in the theory is 2π and the unit of length used in the non-dimensionalization is the healing length a, which in applications is typically a few ångströms.

It is natural to the fluid mechanicist to consider the oscillations of the vortex lines in the condensate as a perturbation to the potential and density in the basic equations (2.1) and (2.2). Therefore we set the density ρ equal to R^2 and represent the potential and square-root density by

$$\phi = \phi_s + \phi' \qquad R = R_s + R'$$

where ϕ' and R' are small perturbations to the steady-state potential ϕ_s and density R_s so that second-order quantities in the primed variables may be neglected. Then substitution into (2.1) and (2.2) yields

$$2R_s\frac{\partial R'}{\partial t} = \frac{\partial}{\partial x_i}\left(R_s^2\frac{\partial \phi'}{\partial x_i}\right) + \frac{\partial}{\partial x_i}\left(2R_s\frac{\partial \phi_s}{\partial x_i}R'\right) \tag{2.3}$$

$$2R_s\frac{\partial \phi'}{\partial t} = 2R_s\frac{\partial \phi_s}{\partial x_i}\frac{\partial \phi'}{\partial x_i} + 2R_s^2R' - \frac{\partial^2 R'}{\partial x_i\partial x_i} + \frac{R'}{R_s}\frac{\partial^2 R_s}{\partial x_i\partial x_i} \tag{2.4}$$

where the repeated suffix denotes summation over $i = 1, 2$, and 3. The variables ϕ_s and R_s satisfy the steady-state equations

$$\frac{\partial}{\partial x_i}\left(R_s^2\frac{\partial \phi_s}{\partial x_i}\right) = 0 \tag{2.5}$$

$$R_s^2 - 1 - \frac{1}{R_s}\frac{\partial^2 R_s}{\partial x_i\partial x_i} = -\left(\frac{\partial \phi_s}{\partial x_i}\right)^2. \tag{2.6}$$

From these equations we can deduce an interesting integral relationship which will later become important in our determination of the frequency of the oscillating vortex lines. First we manipulate equations (2.3) to (2.6) and then integrate over any fixed simply connected region, of volume V and enclosed by a surface S, in which ϕ_s, R_s, ϕ', R' and their derivatives are finite and single valued. We substitute for $\nabla^2 R_s/R_s$ from (2.6) into (2.4), multiply (2.4) by $\partial R_s/\partial x_j$ and also multiply (2.3) by $\partial \phi_s/\partial x_j$ and subtract this from (2.4). Then, on integrating over the volume V and using the divergence theorem, integration by parts and equation (2.5), we eventually obtain in vector notation

$$\begin{aligned}2\int_v \frac{\partial}{\partial t}(R_s\nabla R_s\phi' - R_s\nabla\phi_s R')\,\mathrm{d}V &= \int_s R'(\mathrm{d}\boldsymbol{s}.\nabla)\nabla R_s - \int_s \nabla R_s(\nabla R'.\mathrm{d}\boldsymbol{s}) \\ &- \int_s 2R_sR'\nabla\phi_s(\nabla\phi_s.\mathrm{d}\boldsymbol{s}) - \int_s R_s^2\nabla\phi_s(\nabla\phi'.\mathrm{d}\boldsymbol{s}) + \int_s R_s^2(\nabla\phi_s.\nabla\phi')\,\mathrm{d}\boldsymbol{s} \\ &- \int_s R_s^2\nabla\phi'(\nabla\phi_s.\mathrm{d}\boldsymbol{s}).\end{aligned} \tag{2.7}$$

The possible existence of this relationship was suggested by Professor P. H. Roberts after an analogous result (equation (3.16))—which can be regarded as a particular application of (2.7)—had been obtained in the following analysis of the rectilinear vortex oscillations.

3. The oscillations of the rectilinear vortex

We examine the form of the pertubation equations (2.3) and (2.4) for the rectilinear vortex using cylindrical polar coordinates (r, θ, z). The steady-state solution $R_s = R_0$ and $\phi_s = \phi_0$ appropriate to a rectilinear vortex is given by Ginzburg and Pitaevskii (1958) as

$$\phi_0 = \theta \qquad \frac{\mathrm{d}^2R_0}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}R_0}{\mathrm{d}r} = R_0\left(R_0^2 - 1 + \frac{1}{r^2}\right) \tag{3.1}$$

satisfying $R_0(\infty) = 1$. On substituting (3.1) into (2.3) and (2.4) we find

$$2\frac{\partial R'}{\partial t} = R_0\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2}\right)\phi'+2\frac{\mathrm{d}R_0}{\mathrm{d}r}\frac{\partial\phi'}{\partial r}+\frac{2}{r^2}\frac{\partial R'}{\partial\theta} \tag{3.2a}$$

$$2R_0\frac{\partial\phi'}{\partial t} = 2\frac{R_0}{r^2}\frac{\partial\phi'}{\partial\theta}+R'\left(3R_0{}^2-1+\frac{1}{r^2}\right)-\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2}\right)R'. \tag{3.2b}$$

If we now look for solutions of the form

$$\phi' = \phi(r)\sin(m\theta+kz-\omega t) \qquad R' = R(r)\cos(m\theta+kz-\omega t) \tag{3.3}$$

equations (3.2) yield

$$2\left(\omega+\frac{m}{r^2}\right)R = R_0\left\{\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}-\left(k^2+\frac{m^2}{r^2}\right)\right\}\phi+2\frac{\mathrm{d}R_0}{\mathrm{d}r}\frac{\mathrm{d}\phi}{\mathrm{d}r} \tag{3.4a}$$

$$2\left(\omega+\frac{m}{r^2}\right)R_0\phi = \left\{\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}-\left(k^2+\frac{m^2}{r^2}\right)\right\}R-\left(3R_0{}^2-1+\frac{1}{r^2}\right)R. \tag{3.4b}$$

From these equations we can derive the eigenvalue equations in the form obtained by Pitaevskii (1961) and Fetter (1965). If we let $\psi = R_0\phi$ and introduce variables u and v such that $u = R+\psi$, $v = R-\psi$ and then add and subtract (3.4b) and (3.4a), we obtain

$$\frac{\mathrm{d}^2u}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}u}{\mathrm{d}r}-\left(\frac{(1+m)^2}{r^2}+k^2-1+2R_0{}^2\right)u-R_0{}^2v = 2\omega u \tag{3.5a}$$

$$\frac{\mathrm{d}^2v}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}v}{\mathrm{d}r}-\left(\frac{(1-m)^2}{r^2}+k^2-1+2R_0{}^2\right)v-R_0{}^2u = -2\omega v \tag{3.5b}$$

which are precisely the equations referred to above. We have the physical interpretation that $(u+v)/2 = R$ and $(u-v)/2R_0 = \phi$ are the perturbations to the density and potential respectively.

Equations (3.5a) and (3.5b) are unaltered by either of the transformations $(k\leftrightarrow -k)$ or $(u\leftrightarrow v,\ \omega\leftrightarrow -\omega$ and $m\leftrightarrow -m)$. In fact if

$$\phi' = \phi(r)\sin(m\theta+kz-\omega t)$$

is the perturbation to the potential due to the oscillation, the only other independent solution with sine dependence is $\phi' = \phi(r)\sin(m\theta-kz-\omega t)$. Thus two possible solutions for ϕ' are of the form

$$\phi' = \phi(r)\cos kz\sin(m\theta-\omega t) \qquad \phi' = \phi(r)\sin kz\cos(m\theta-\omega t).$$

We see that we are forced to look for travelling-wave solutions and in every case at a given height z the waves of constant disturbance proceed around the core in a direction opposite to the circulation of the vortex itself.

We now examine the solutions to (3.5a) and (3.5b) for small r and large r. The equations have a regular singularity at the origin and as discussed by Fetter (1965) there are four linearly independent solutions valid there. It is not difficult to show that these four solutions have the following behaviour:

$$\begin{array}{ll} u\propto r^{m+3},\ v\propto r^{m-1} & u\propto r^{m+1},\ v\propto r^{m+5} \\ u\propto r^{5-m},\ v\propto r^{1-m} & u\propto r^{-(1+m)},\ v\propto r^{3-m}. \end{array} \tag{3.6}$$

For large r the equations (3.5a) and (3.5b) become

$$\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}-(k^2+1+2\omega)u-v = 0 \qquad (3.7a)$$

$$\frac{d^2v}{dr^2}+\frac{1}{r}\frac{dv}{dr}-(k^2+1-2\omega)v-u = 0 \qquad (3.7b)$$

where terms in r^{-2} have been neglected. These equations are independent of m so that the asymptotic behaviour for large r is the same for all m. The solutions to (3.7) are

$$u,\, v \propto r^{-1/2}\, e^{\beta r}$$

where

$$\{\beta^2-(k^2+1)\}^2-(1+4\omega^2)=0$$

so that β is given by one of the relations

$$\beta_1{}^2 = k^2+1+(1+4\omega^2)^{1/2} \qquad \beta_2{}^2 = k^2+1-(1+4\omega^2)^{1/2}.$$

For real values of the wavelength and frequency, $\beta_1{}^2$ is always positive and there are two solutions which behave like $r^{-1/2}\exp(\pm\beta_1 r)$. If in addition $4\omega^2 < k^2(k^2+2)$, then $\beta_2{}^2$ is positive and the other two solutions behave like $r^{-1/2}\exp(\pm\beta_2 r)$. When, however, $4\omega^2 > k^2(k^2+2)$, $\beta_2{}^2$ is negative and equal to $-\alpha^2$ say, and the two remaining solutions behave like $r^{-1/2}\exp(\pm \mathrm{i}\alpha r)$.

Each of the solutions (3.6) at the origin is, when followed across the interval $0 < r < \infty$, a linear combination of all four of the solutions regular at infinity. In either of the cases $4\omega^2 \gtrless k^2(k^2+2)$ the behaviour of each of the solutions at the origin is dominated by the $r^{-1/2}\exp(\beta_1 r)$ term at infinity. If, however, we combine the solutions in (3.6), finite at the origin, in the correct ratio, we can cancel this growing exponential term. When $4\omega^2 > k^2(k^2+2)$ the solutions will then behave like

$$u,\, v \propto r^{-1/2}\{a_1\exp(-\beta_1 r)+b_1\exp(-\mathrm{i}\,\alpha r)+c_1\exp(\mathrm{i}\,\alpha r)\}. \qquad (3.8)$$

The solution is oscillatory and decaying and there is a continuum of eigenvalues. In the case when $4\omega^2 < k^2(k^2+2)$ the solution will behave like

$$u,\, v \propto r^{-1/2}\{a_2\exp(-\beta_1 r)+b_2\exp(-\beta_2 r)+c_2\exp(\beta_2 r)\}. \qquad (3.9)$$

In general the constant c_2 in (3.9) will depend on m, k, and ω, and for a solution bounded at infinity we require that $c_2(m, k, \omega) = 0$. If there are such roots to this equation then the solution in (3.9) will decay exponentially and the eigenvalues will show up as bound states below the edge, $4\omega^2 = k^2(k^2+2)$, of the continuum region. This has been discussed previously by Fetter (1965).

The special case $m = 1$

In the case $m = 1$ it follows from (3.6) that since, for small r, $R_0 \sim k'r$ (where k' is a constant: I equation (3.14)), the radial parts of the acceptable solutions for the perturbations ϕ' and R' behave as

$$\phi \propto r \qquad R \propto r^2 \qquad (3.10)$$

$$\phi \propto \frac{1}{r} \qquad R \propto 1. \qquad (3.11)$$

These two solutions in this special case are easily interpreted. Consider first (3.10) which corresponds by (3.3) to $\phi' = \epsilon r \sin(\theta + kz - \omega t)$ for small r. In the long-wavelength limit $k \to 0$ the velocity $-\nabla\phi'$ due to this perturbation is

$$\{\epsilon \sin(\theta + kz - \omega t),\ \epsilon \cos(\theta + kz - \omega t),\ 0\}.$$

This mode may be interpreted as a physical displacement of the vortex axis which moves in a circular orbit in an opposite sense to the circulation. In such a displacement we can show that there are perturbations to ϕ and R of the form (3.11).

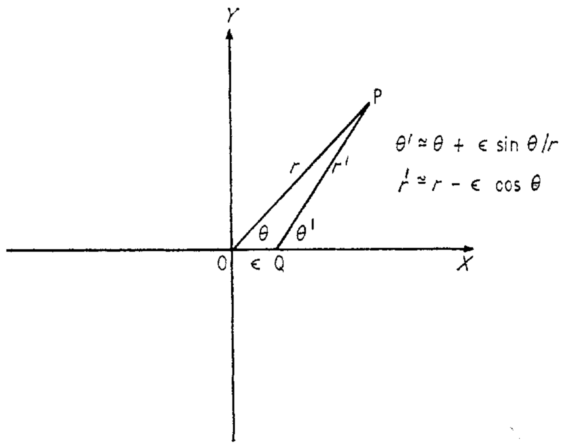

Figure 1. Indicating the new potential and density at a point P when the vortex axis is moved an arbitrarily small distance ϵ from O to Q.

We consider a cross section of a rectilinear vortex whose axis is along Oz. In the (r, θ) plane shown the axis is at O and the potential and density at P are given by $\phi_P = \theta$ and $R_P = R_0(r)$. If the axis is displaced an arbitrarily small distance ϵ to a point Q, the new potential and density at P are $\phi_{pn} = \theta'$ and $R_{pn} = R_0(r')$. The perturbations to the potential and density are therefore

$$\phi' = \phi_{pn} - \phi = \frac{\epsilon \sin\theta}{r} \qquad R' = R_{pn} - R = -\epsilon \cos\theta \frac{\mathrm{d}R_0}{\mathrm{d}r}$$

and for $r \to 0$ the radial parts of the perturbation functions are of the form (3.11). We note here that, for $k = 0$, there are in fact exact solutions for ϕ and R

$$\phi = \frac{1}{r} \qquad R = -\frac{\mathrm{d}R_0}{\mathrm{d}r} \tag{3.12}$$

with the eigenvalue $\omega = 0$, which corresponds to the displacement just considered. This can be verified by directly substituting $m = 1$, $k = 0$ and $\omega = 0$ into (3.4).

Backed by this physical insight of the $m = 1$ mode, we can now proceed to compute the frequency ω of the 'long-wavelength' oscillations. We examine equations (3.4) for $m = 1$, in the limit $k \to 0$. The method we use to solve these equations is that of 'matched asymptotic expansions'. The solution is first sought in an 'inner' cylindrical region, of radius r, centred on the Oz axis, as an expansion about the exact solution (3.12). The resulting solution is then examined for $r \to \infty$. In this limit nonuniformities arise and so the solution to (3.4) is also found in an 'exterior' region characterized by a new stretched coordinate $s = kr$, which is of order unity in the region of nonuniformity. This solution is examined in the limit $s \to 0$ and finally the two asymptotic solutions are matched in an 'overlap domain', where both the inner and outer expansions are valid.

Interior solution

For the inner expansion we express R and ϕ by writing

$$\phi = \frac{1}{r} + k^2\phi_1 + \ldots \qquad R = -\frac{\mathrm{d}R_0}{\mathrm{d}r} + k^2 R_1 + \ldots \tag{3.13}$$

while we also write ω in the form $\omega = \omega_1 k^2$. We justify this last step by equation (3.18) in the subsequent analysis, though in any case it is apparent when we compare with (1.4) and (1.5). Then on substituting (3.13) into (3.4) and equating coefficients of k^2 we obtain

$$R_0\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{1}{r^2}\right)\phi_1 + 2\frac{\mathrm{d}R_0}{\mathrm{d}r}\frac{\mathrm{d}\phi_1}{\mathrm{d}r} - \frac{2R_1}{r^2} = -2\omega_1\frac{\mathrm{d}R_0}{\mathrm{d}r} + \frac{R_0}{r} \tag{3.14a}$$

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{1}{r^2}\right)R_1 - \left(3R_0^2 - 1 + \frac{1}{r^2}\right)R_1 - 2R_0\frac{\phi_1}{r^2} = 2\omega_1\frac{R_0}{r} - \frac{\mathrm{d}R_0}{\mathrm{d}r}. \tag{3.14b}$$

After substituting for R_0 for large r from I (equation (3.15)) we find that the complete solutions are of the form

$$\phi_1 = \tfrac{1}{2}r\ln r + A_1 r + \frac{(\ln r)^2}{2r} + \frac{\ln r}{r}(2A_1 + 2\omega_1 + 1) + \frac{A_2}{r} + \ldots \tag{3.15a}$$

$$R_1 = -\frac{1}{2r}\ln r - \frac{A_1}{r} - \frac{\omega_1}{r} + \ldots. \tag{3.15b}$$

Here A_1 and A_2 are constants which multiply the complementary functions and terms involving them must be matched with corresponding expressions in the expansion of the outer solution about $s = 0$.

Equations (3.14) have an interesting property which is important to us in determining ω. If we multiply (3.14a) by $r\,\mathrm{d}R_0/\mathrm{d}r$ and (3.14b) by R_0 and then subtract and rearrange using (3.1), we have

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}R_0}{\mathrm{d}r}\frac{\mathrm{d}R_1}{\mathrm{d}r} - r\frac{\mathrm{d}^2R_0}{\mathrm{d}r^2}R_1 - R_0^2\frac{\phi_1}{r} - R_0^2\frac{\mathrm{d}\phi_1}{\mathrm{d}r}\right) = 2\omega_1\frac{\mathrm{d}}{\mathrm{d}r}(R_0^2) - \left\{r\left(\frac{\mathrm{d}R_0}{\mathrm{d}r}\right)^2 + \frac{R_0^2}{r}\right\}.$$

On integrating from 0 to ∞ we then obtain

$$\left[r\frac{\mathrm{d}R_0}{\mathrm{d}r}\frac{\mathrm{d}R_1}{\mathrm{d}r} - r\frac{\mathrm{d}^2R_0}{\mathrm{d}r^2}R_1 - R_0^2\frac{\phi_1}{r} - R_0^2\frac{\mathrm{d}\phi_1}{\mathrm{d}r}\right]_0^\infty = 2\omega_1\left[R_0^2\right]_0^\infty - \int_0^\infty\left\{r\left(\frac{\mathrm{d}R_0}{\mathrm{d}r}\right)^2 + \frac{R_0^2}{r}\right\}\mathrm{d}r. \tag{3.16}$$

Substitution of the values of R_1, ϕ_1 and R_0 for large and small r finally gives

$$-(2A_1 + \tfrac{1}{2}) = 2\omega_1 - \left\{\int_0^\infty r\left(\frac{\mathrm{d}R_0}{\mathrm{d}r}\right)^2\mathrm{d}r + ⨍_0^\infty \frac{R_0^2}{r}\,\mathrm{d}r\right\} \tag{3.17}$$

where A_1 is the constant appearing in (3.15) and the bar through the second integral denotes its finite part, for although R_0^2/r behaves like $1/r$ for large r, the divergent logarithmic term is cancelled by a similar term from the left hand side of (3.16).

It is important to note that the integral result (3.16) can be obtained as a special application to the rectilinear vortex, of the general integral relationship deduced in § 2. To show this we need only integrate over the right-cylindrical volume V with axis on Oz and whose cross section is an annulus with radii δ, $\bar{R}$ which has a cut along $\theta = 0$. The ends of the cylinder are the plane surfaces at $z = -\pi/2k$ and $z = \pi/2k$ parallel to the (r, θ) plane. This volume is simply connected and so ϕ_0, R_0, ϕ' and R' are single valued. We use the inner expansion for ϕ' and R' and let $\delta \to 0$ and $\bar{R} \to \infty$ on the inner scale. Then on applying (2.7) over $V(0 \leqslant r \leqslant \infty$, $0 < \theta < 2\pi$ and $-\pi/2k \leqslant z \leqslant \pi/2k)$ and resolving along the constant unit vectors $\mathbf{1}_x$ and $\mathbf{1}_y$, we obtain two identical scalar relations of the form

$$2\omega[R_0^2]_0^\infty = k^2\left(\int_0^\infty \left\{r\left(\frac{dR_0}{dr}\right)^2 + \frac{R_0^2}{r}\right\} dr + \left[r\frac{dR_0}{dr}\frac{dR_1}{dr} - r\frac{d^2R_0}{dr^2}R_1 - R_0^2\frac{d\phi_1}{dr} - R_0^2\frac{\phi_1}{r}\right]_0^\infty\right) \tag{3.18}$$

which reproduces equation (3.16) when ω is represented by $\omega = \omega_1 k^2$.

Exterior solution

For the outer independent variable we use $s = kr$. We expand (3.4) for large r and then replace r by s/k. Then in terms of s the equations become

$$2\omega_1 k^2 R = k^2\left(1-\frac{k^2}{2s^2}\right)\left\{\frac{d^2}{ds^2} + \frac{1}{s}\frac{d}{ds} - \left(1+\frac{1}{s^2}\right)\right\}\phi + 2\frac{k^4}{s^2}\frac{d\phi}{ds} - 2k^2\frac{R}{s^2} + \dots \tag{3.19a}$$

$$2\omega_1 k^2\left(1-\frac{k^2}{2s^2}\right)\phi = k^2\left\{\frac{d^2}{ds^2} + \frac{1}{s}\frac{d}{ds} - \left(1+\frac{1}{s^2}\right)\right\}R - 2\left(1-\frac{k^2}{s^2}\right)R - \frac{k^2}{s^2}\left(2-\frac{k^2}{s^2}\right)\phi\dots \tag{3.19b}$$

On expanding R and ϕ by

$$R = R_{E1} + k^2 R_{E2} + \dots \qquad \phi = \phi_{E1} + k^2\phi_{E2} + \dots \tag{3.20}$$

substituting into (3.19*b*) and equating successive powers of k^2 we obtain

$$R_{E1} = 0 \qquad R_{E2} = -\left(\omega_1 + \frac{1}{s^2}\right)\phi_{E1}. \tag{3.21}$$

From (3.19*a*) we also have

$$\left\{\frac{d^2}{ds^2} + \frac{1}{s}\frac{d}{ds} - \left(1+\frac{1}{s^2}\right)\right\}\phi_{E1} = 0 \tag{3.22a}$$

$$\left\{\frac{d^2}{ds^2} + \frac{1}{s}\frac{d}{ds} - \left(1+\frac{1}{s^2}\right)\right\}\phi_{E2} = 2\left(\omega_1 + \frac{1}{s^2}\right)R_{E2} - \frac{2}{s^3}\frac{d\phi_{E1}}{ds}. \tag{3.22b}$$

The solution to (3.22*a*) bounded at infinity is the modified Bessel function of first order $K_1(s)$ and the expansion of this solution about $s = 0$ gives

$$\phi_{E1} = C\left\{\frac{1}{s} + \frac{s}{2}\ln\left(\frac{s}{2}\right) - \frac{s}{4}(1-2\gamma) + \frac{s^2}{16}\ln\left(\frac{s}{2}\right) + \dots\right\} \tag{3.23}$$

where C is a constant. Then from (3.21) and (3.22*b*) we find

$$R_{E2} = -C\left\{\frac{1}{s^3}+\frac{1}{2s}\ln\left(\frac{s}{2}\right)+\frac{1}{s}\left(\omega_1+\frac{\gamma}{2}-\tfrac{1}{4}\right)+\tfrac{1}{16}\ln\left(\frac{s}{2}\right)+\ldots\right\} \quad (3.24a)$$

$$\phi_{E2} = C\left\{\frac{1}{2s}\left(\ln\frac{s}{2}\right)^2+\left(\frac{\ln s}{s}\right)(2\omega_1+\gamma+\tfrac{1}{2})+\ldots\right\}. \quad (3.24b)$$

Matching

The inner solution for $r \to \infty$ must match the outer solution for $s \to 0$ to all terms. The process fixes all unknown constants and gives a consistent solution. To express the inner solution for large r in terms of s we replace r by s/k and $\ln r$ by $\ln s - \ln k$ in (3.15). We obtain

$$\phi = k\left\{\frac{1}{s}+\frac{s}{2}\ln s+s(A_1-\tfrac{1}{2}\ln k)+\ldots\right\}+k^3\left\{\frac{(\ln s)^2}{2s}+\frac{\ln s}{s}(-\ln k+2A_1 +2\omega_1+1)+\ldots\right\} \quad (3.25a)$$

$$R = -k^3\left\{\frac{1}{s^3}+\frac{\ln s}{2s}-\frac{1}{s}(A_1-\tfrac{1}{2}\ln k+\omega_1)+\ldots\right\}. \quad (3.25b)$$

A comparison of (3.25) with the corresponding terms in the exterior solution for small s from (3.23) and (3.24) consistently shows that

$$C = k \qquad 2A_1+\tfrac{1}{2} = -\left(\ln\frac{2}{k}-\gamma\right).$$

Then on substituting into (3.17) we obtain

$$\omega = \frac{k^2}{2}\left(\ln\frac{2}{k}-\gamma+\epsilon_c-\tfrac{1}{2}\right) \quad (3.26)$$

where ϵ_c is the dimensionless energy per unit length of the 'core' of the vortex and is given by (1.8). In terms of our physical variables this result finally becomes (1.7)

The determination of the complete spectrum for ω for arbitrary wavelengths requires computational work. For this purpose it is convenient to utilize the eigenvalue equations (3.5) with u and v as the dependent variables, R_0 having been previously determined from (3.1). The analysis between (3.5) and (3.9) then applies. Bound states with the corresponding exponentially decaying eigenfunction solutions were obtained in the cases $m = +1, -1, 0$ and 2. The eigenvalue in the case $m = 1$ is shown graphically in figure 2 for the range $0 < k < 0{\cdot}12$, together with the theoretical eigenvalue valid for the long-wavelength limit $k \to 0$. The agreement is close, the discrepancy at the origin, where a broken curve is shown, being due to numerical difficulties. In figure 3 the bound state $m = 1$ is drawn, for the range $0 < k < 2{\cdot}5$, below the edge of the continuum region. The bound state branches for $m = 0$ and $m = 2$ are so close to the continuum region that they are omitted from the figure for clarity.

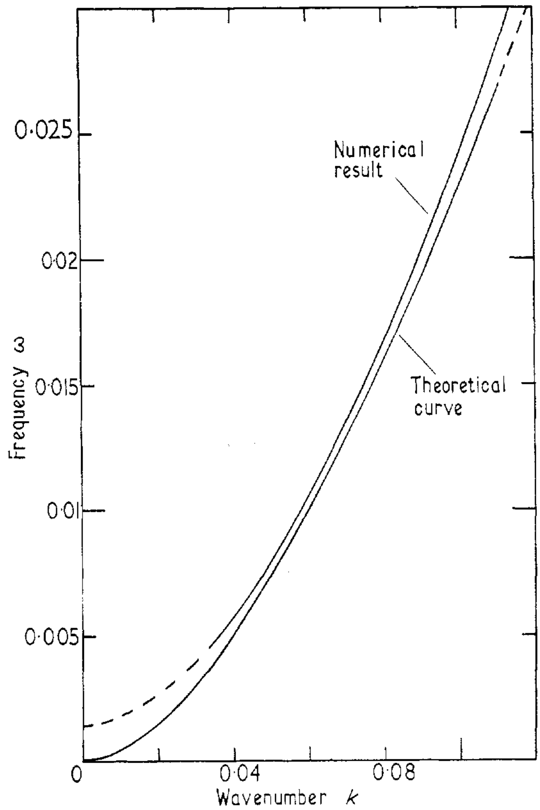

Figure 2. Comparison of the eigenvalue obtained numerically from (3.5) for the bound state $m = 1$ for small k, with the theoretical value given by (3.26).

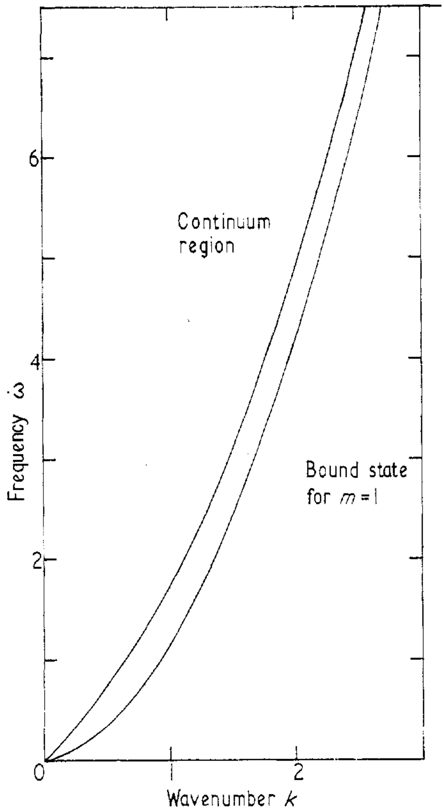

Figure 3. The bound state $m = 1$ below the edge $\omega^2 = k^2(k^2+2)$ of the continuum region.

4. The oscillations of the large circular vortex

An asymptotic analysis similar to that developed for the rectilinear vortex equations can be employed to solve the perturbation equations (2.3) and (2.4) for the large circular vortex whose radius, in nondimensional units, is c, where $c \gg 1$. The frame of reference adopted is that with which Roberts and Grant have recently investigated the structure of the vortex in its steady state, that is, one in which the vortex is at rest and the fluid at infinity is moving with velocity $U_0(c)$. If (ω, θ, z) are cylindrical coordinates as shown in figure 4, the singularity of curl $\boldsymbol{u}$ is on the circle VV$'$($z = 0$, $\tilde{\omega} = c$) and Oz is the axis of symmetry. Coordinates (T, χ, θ), often referred to as displaced polars, are also introduced where

$$\tilde{\omega} = c - T\cos\chi \qquad z = T\sin\chi. \tag{4.1}$$

The distance V$'$P $\equiv T_1$ is given by

$$T_1^2 = 4c^2 - 4cT\cos\chi + T^2. \tag{4.2}$$

Again we examine the perturbation equations, in the limit $c \to \infty$, using 'inner' and 'outer' expansions. First the solution is found in the inner toroidal region centred on VV$'$ whose radius is T and then this is examined in the limit $T \to \infty$.

For the 'outer region' we use the stretched coordinate $s = T/c$ and consider the solution as $s \to 0$. Finally, the two asymptotic solutions are matched in the 'overlap domain' (where both the inner and outer expansions are valid).

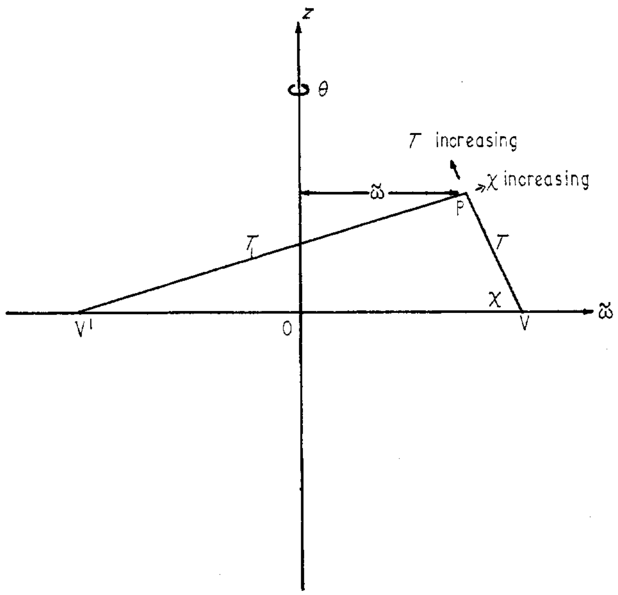

Figure 4. Illustrating the circular vortex and its related coordinate systems: $(\tilde{\omega}, \theta, z)$ are cylindrical coordinates, (T, χ, θ) are displaced polar coordinates. The singularity associated with the vortex line is indicated by V and V′ in the cross section.

Inner solution

The steady state solution for R_s in the inner region has been given in I in powers of $1/c$ as far as the second term. For this analysis we require an extra term and therefore we set

$$R_s = R_0(T) + \frac{1}{c} R_1(T) \cos\chi + \frac{1}{c^2}(Q_0(T) + Q_2(T)\cos 2\chi). \tag{4.3}$$

The solutions for R_0 and R_1 for large T are given in I (equations (3.15) and (3.23)) and in addition we find that for large T

$$Q_0 = -\frac{(\ln T)^2}{8} + \frac{\ln T}{8}(4B+1) - \left(\frac{B^2}{2} + \frac{B}{4} + \tfrac{1}{16}\right) + \dots \tag{4.4a}$$

$$Q_2 = \frac{\ln T}{2} - \tfrac{1}{4}(8F+3B+1) - \frac{(\ln T)^3}{4T^2} + \dots \tag{4.4b}$$

where B is a constant given in I equation (3.27), $F = (3\lambda - 4B + 2)/16$ and $\lambda = \ln(8c) - 2$. Similarly we represent the stream function ψ_s for the flow by (see I equation (3.2))

$$\psi_s = c\psi_0(T) + \psi_1(T)\cos\chi + \frac{1}{c}(P_0(T) + P_2(T)\cos 2\chi) \tag{4.5}$$

and for large T

$$P_0 = \frac{T^2}{8}\ln T - \frac{T^2}{16}(4B+3) + \dots \tag{4.6a}$$

$$P_2 = -\frac{T^2}{16}\ln T + FT^2 - \ln T + \dots . \tag{4.6b}$$

In terms of this stream function the velocity is given by

$$\boldsymbol{u}_s = \frac{1}{R_s^2(c - T\cos\chi)}\left(-\frac{1}{T}\frac{\partial\psi_s}{\partial\chi}, \frac{\partial\psi_s}{\partial T}, 0\right) \tag{4.7}$$

and hence we obtain

$$\boldsymbol{u}_s = \left(0, \frac{1}{T}, 0\right) + \frac{1}{c}\left(\frac{\psi_1}{R_0^2 T}\sin\chi, Z_0\cos\chi, 0\right) + \frac{1}{c^2}(Z_1\sin 2\chi, Z_2 + Z_3\cos 2\chi, 0)$$

where

$$Z_0 = 1 - \frac{2R_1}{R_0 T} + \frac{\psi_1'}{R_0^2}, \qquad Z_2 = \frac{P_0'}{R_0^2} + \frac{T\psi_1'}{2R_0^2} + \frac{T}{2} - \frac{R_1\psi_1'}{R_0^3} - \frac{R_1}{R_0} - \frac{2Q_0}{R_0 T} + \frac{3R_1^2}{2R_0^2 T} \tag{4.8a}$$

$$Z_1 = \frac{2P_2}{R_0^2 T} + \frac{\psi_1}{2R_0^2} - \frac{R_1\psi_1}{R_0^3 T} \qquad Z_3 = \frac{P_2'}{R_0^2} + \frac{T\psi_1'}{2R_0^2} + \frac{T}{2} - \frac{R_1\psi_1'}{R_0^3} - \frac{R_1}{R_0} - \frac{2Q_2}{R_0 T} + \frac{3R_1^2}{2R_0^2 T}. \tag{4.8b}$$

The steady-state potential ϕ_s is given by

$$\phi_s = -\chi + \frac{1}{c}G(T)\sin\chi + \frac{1}{c^2}H(T)\sin 2\chi \tag{4.9}$$

where

$$\frac{dG}{dT} = -\frac{\psi_1}{R_0^2 T} \qquad \frac{G}{T} = -Z_0 \tag{4.10a}$$

$$\frac{dH}{dT} = -Z_1 \qquad \frac{H}{T} = -\tfrac{1}{2}Z_3. \tag{4.10b}$$

The solutions of the perturbation equations for the large circular vortex resemble, in the limit $c \to \infty$, the solutions for the rectilinear vortex obtained in § 3, and again we can find the inner expansion by iterating about the exact solutions whose physical significance we have discussed. In this case, however, we look for solutions of the form

$$\phi' = \cos p\theta\left\{\left(\alpha_p\frac{\sin\chi}{T} + \gamma_p\frac{\cos\chi}{T}\right) + \frac{1}{c}\{\alpha_p J\sin\chi + \gamma_p(K + L\cos 2\chi)\} + \frac{1}{c^2}\{\alpha_p(M\sin\chi + N\sin 3\chi) + \gamma_p(P\cos\chi + Q\cos 3\chi)\}\right\} \tag{4.11a}$$

$$R' = \cos p\theta \left\{ \left(\alpha_p \frac{dR_0}{dT} \cos\chi - \gamma_p \frac{dR_0}{dT} \sin\chi \right) + \frac{1}{c} \{\alpha_p (R + S\cos 2\chi) - \gamma_p U \sin 2\chi\} + \frac{1}{c^2} \{\alpha_p (V\cos\chi + W\cos 3\chi) - \gamma_p (X\sin\chi + Y\sin 3\chi)\} \right\}. \quad (4.11b)$$

Here J, K, L, M, N, P, Q, R, S, U, V, W, X and Y are all functions of T, and α_p and γ_p are functions of time which have different amplitudes ϵ_1 and ϵ_2. From a subsequent application of (2.7) we will find that we must take α_p and γ_p as

$$\alpha_p = \epsilon_1 \cos(\omega t + \mu) \qquad \gamma_p = \epsilon_2 \sin(\omega t + \mu) \quad (4.12)$$

where ω can be written in the form $\omega = \omega_2/c^2$ and μ is an arbitrary phase angle. It is to be noted that (even though in the rectilinear case we were compelled to consider a progressive wave solution) when a similar solution of the form

$$\phi' = \epsilon_1 \cos p\theta \left[\left\{ \frac{1}{T} \sin(\chi + \omega t + \mu) + \ldots \right\} \right]$$

$$R' = \epsilon_1 \cos p\theta \left[\left\{ \frac{dR_0}{dT} \cos(\chi + \omega t + \mu) + \ldots \right\} \right]$$

was taken for the circular vortex (i.e. $\epsilon_1 = \epsilon_2$ above), this eventually led to two contradictory values for ω. The significant point that the more general form, given by (4.11) and (4.12) with ϵ_1 and ϵ_2 different, should be considered, was a suggestion of Professor P. H. Roberts. It will transpire that when the integer p, which represents the number of waves around the circumference of the ring, is large then $\epsilon_1 - \epsilon_2 \sim O(1/p^2)$. In the general case this is not so.

The solutions (4.11) represent the perturbations to the density and potential of a vortex ring which is disturbed very slightly from its circular form in the $z = 0$ plane and whose initial configuration is given by

$$R_c = c + \alpha_p \cos p\theta \quad (4.13a)$$

$$Z_c = \gamma_p \cos p\theta \quad (4.13b)$$

where R_c and Z_c are the radial distance of the singularity of the vortex from the axis Oz and its height above the $z = 0$ plane. For the displacement (4.13a) we find that

$$\chi \to \chi - \alpha_p \frac{\sin\chi}{T} \cos p\theta \qquad T \to T + \alpha_p \cos\chi \cos p\theta$$

and the new density and potential are given by substituting these expressions into (4.3) and (4.9) and neglecting second-order terms in α_p and γ_p. On subtracting the old potential and density we obtain ϕ' and R'. Similarly, for the disturbance (4.13b), we find

$$\chi \to \chi - \gamma_p \frac{\cos\chi}{T} \cos p\theta \qquad T \to T - \gamma_p \sin\chi \cos p\theta$$

from which we can compute ϕ' and R'. The combined effect of both displacements is to give rise to the perturbations in (4.11).

We now substitute from (4.11) into the perturbation equations (2.3) and (2.4) with the appropriate expressions for the operators ∇^2 and ∇ (cf. I equations (2.44), (2.45) and (3.1)). On equating coefficients of $c^{-1}\alpha_p \cos p\theta$ and $c^{-1}\alpha_p \cos 2\chi \cos p\theta$ in (2.4) and expanding for large T we obtain

$$-2R+\left(\frac{\mathrm{d}^2}{\mathrm{d}T^2}+\frac{1}{T}\frac{\mathrm{d}}{\mathrm{d}T}+\frac{2}{T^2}\right)R+\ldots = -\frac{1}{2T^2}+\frac{(\ln T)^2}{T^4}-\tfrac{1}{2}(8B+1)\frac{\ln T}{T^4}+\ldots \tag{4.14a}$$

$$-2S+\frac{4J}{T^2}+\left(\frac{\mathrm{d}^2}{\mathrm{d}T^2}+\frac{1}{T}\frac{\mathrm{d}}{\mathrm{d}T}-\frac{2}{T^2}\right)S+\ldots = \frac{\ln T}{T^2}-\frac{1}{2T^2}(4B+1)+\frac{\ln T}{4T^4}+\ldots \tag{4.14b}$$

while on equating coefficients of $c^{-1}\gamma_p \sin 2\chi \cos p\theta$ and expanding we have an equation for U and L which is identical to (4.14*b*) in S and J. Similarly, after equating coefficients of $c^{-1}\gamma_p \cos p\theta$, $c^{-1}\alpha_p \sin 2\chi \cos p\theta$ and $c^{-1}\gamma_p \cos 2\chi \cos p\theta$, for large T (2.3) yields

$$\frac{\mathrm{d}K}{\mathrm{d}T} = -\frac{1}{2T}+\frac{\ln T}{T^3}-\frac{1}{2T^3}(4B+1)+\frac{4\ln T}{T^5}+\ldots \tag{4.15a}$$

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}T^2}+\frac{1}{T}\frac{\mathrm{d}}{\mathrm{d}T}-\frac{4}{T^2}\right)J+\left(\frac{2}{T^3}\frac{\mathrm{d}J}{\mathrm{d}T}+\frac{4S}{T^4}\right)+\ldots = -\frac{1}{T^2}+2\frac{\ln T}{T^4}-\frac{4B}{T^4}-2\frac{(\ln T)^2}{T^6}+\ldots \tag{4.15b}$$

and again an equation for U and L identical to (4.15*b*) in S and J. On integrating (4.15*a*) we find

$$K = -\tfrac{1}{2}\ln T+\alpha-\frac{\ln T}{2T^2}+\frac{B}{T^2}-\frac{\ln T}{T^4}+\ldots \tag{4.16}$$

where α is a constant of integration, and (4.14*a*) gives

$$R = \frac{1}{4T^2}-\frac{(\ln T)^2}{2T^4}+\tfrac{1}{4}(8B+1)\frac{\ln T}{T^4}+\ldots. \tag{4.17}$$

The pair of equations (4.14*b*) and (4.15*b*) yield

$$J = L = \tfrac{1}{4}-\frac{(\ln T)^2}{2T^2}+\tfrac{1}{2}(4B+1)\frac{\ln T}{T^2}+\ldots \tag{4.18}$$

$$S = U = -\frac{\ln T}{2T^2}+\frac{1}{4T^2}(4B+3)+\ldots. \tag{4.19}$$

The second-order equations, obtained from (2.3) by equating coefficients of

$c^{-2}\alpha_p \cos\chi \cos p\theta$, $c^{-2}\alpha_p \cos 3\chi \cos p\theta$, $c^{-2}\alpha_p \sin\chi \cos p\theta$ and $c^{-2}\gamma_p \sin 3\chi \cos p\theta$

respectively and expanding for large T, are

$$\left(-2V+\frac{2M}{T^2}\right)+\ldots = \frac{\ln T}{4\,T}-\frac{1}{T}\left(\tfrac{1}{2}B+\tfrac{7}{16}+2\frac{\epsilon_2}{\epsilon_1}\,\omega_2\right)+\mathrm{O}\left(\frac{(\ln T)^2}{T^3}\right) \tag{4.20a}$$

$$\left(-2W+\frac{6N}{T^2}\right)+\ldots = \frac{\ln T}{T}-\frac{1}{T}(4F+\tfrac{3}{2}B+\tfrac{7}{16})+\mathrm{O}\left(\frac{(\ln T)^2}{T^3}\right) \tag{4.20b}$$

$$\left(-2X+\frac{2P}{T^2}\right)+\ldots = \tfrac{3}{4}\frac{\ln T}{T}-\frac{1}{T}\left(\tfrac{3}{2}B+\tfrac{1}{16}+2\frac{\epsilon_1}{\epsilon_2}\,\omega_2\right)+\mathrm{O}\left(\frac{(\ln T)^2}{T^3}\right) \tag{4.20c}$$

and an equation for Y and Q identical to (4.20b) in W and N. In addition after equating coefficients of

$$c^{-2}\alpha_p \sin\chi \cos p\theta,\ c^{-2}\alpha_p \sin 3\chi \cos p\theta,\ c^{-2}\gamma_p \cos\chi \cos p\theta \text{ and } c^{-2}\gamma_p \cos 3\chi \cos p\theta$$

in (2.4) for large T, we have

$$\left\{\frac{1}{T}\frac{\mathrm{d}}{\mathrm{d}T}\left(T\frac{\mathrm{d}M}{\mathrm{d}T}\right) - \frac{M}{T^2}\right\} + \ldots = \frac{1}{T}(p^2-\tfrac{1}{4}) + \mathrm{O}\left(\frac{(\ln T)^2}{T^3}\right) \tag{4.21a}$$

$$\left\{\frac{1}{T}\frac{\mathrm{d}}{\mathrm{d}T}\left(T\frac{\mathrm{d}N}{\mathrm{d}T}\right) - \frac{9N}{T^2}\right\} + \ldots = -\frac{3}{4T} + \mathrm{O}\left(\frac{(\ln T)^2}{T^3}\right) \tag{4.21b}$$

$$\left\{\frac{1}{T}\frac{\mathrm{d}}{\mathrm{d}T}\left(T\frac{\mathrm{d}P}{\mathrm{d}T}\right) - \frac{P}{T^2}\right\} + \ldots = \frac{1}{T}(p^2-\tfrac{3}{4}) + \mathrm{O}\left(\frac{(\ln T)^2}{T^3}\right). \tag{4.21c}$$

The equation for Q is the same as (4.21b) in N. These three sets of equations have solutions of the form

$$M = \tfrac{1}{2}(p^2-\tfrac{1}{4})T\ln T + D_1 T + \mathrm{O}\left(\frac{(\ln T)^3}{T}\right)$$

$$V = \tfrac{1}{2}(p^2-\tfrac{1}{2})\frac{\ln T}{T} + \frac{1}{T}\left(D_1 + \frac{B}{4} + \tfrac{7}{32} + \frac{\epsilon_2}{\epsilon_1}\omega_2\right) + \mathrm{O}\left(\frac{(\ln T)^3}{T^3}\right) \tag{4.22a}$$

$$N = Q = \frac{3T}{32} + \mathrm{O}\left(\frac{(\ln T)^2}{T}\right)$$

$$W = Y = -\tfrac{1}{2}\frac{\ln T}{T} + \frac{1}{4T}(3B + 8F + 2) + \mathrm{O}\left(\frac{(\ln T)^2}{T^3}\right) \tag{4.22b}$$

$$P = \tfrac{1}{2}(p^2-\tfrac{3}{4})T\ln T + D_2 T + \mathrm{O}\left(\frac{(\ln T)^3}{T}\right)$$

$$X = \tfrac{1}{2}(p^2-\tfrac{3}{2})\frac{\ln T}{T} + \frac{1}{T}\left(D_2 + \frac{3B}{4} + \tfrac{1}{32} + \frac{\epsilon_1}{\epsilon_2}\omega_2\right) + \mathrm{O}\left(\frac{(\ln T)^3}{T^3}\right) \tag{4.22c}$$

where D_1 and D_2 are constants which multiply the complementary function solutions.

Having obtained this inner solution we apply the integral relationship (2.7) directly to the circular vortex to give a result analogous to (3.18), obtained for the rectilinear vortex. In this case we integrate over the simply connected toroidal volume V with axis VV′ and whose cross section is an annulus of radii δ' and T' with a cut along $\chi = 0$. The ends of the toroid are the plane surfaces at $\theta = -\pi/2p$ and $\theta = \pi/2p$. Again we use the interior expansion for ϕ' and R' when (2.7) is applied and then let $\delta \to 0$ and $T' \to \infty$ on the inner scale. Resolving along the constant vectors $\mathbf{1}_x$ and $\mathbf{1}_z$ we obtain the following two scalar relations:

$$-2\frac{\mathrm{d}\gamma_p}{\mathrm{d}t} = \frac{\alpha_p}{c^2}\left[[G_1]_0^\infty - (p^2-1)\left\{\int_0^\infty r\left(\frac{\mathrm{d}R_0}{\mathrm{d}r}\right)^2 \mathrm{d}r + \int_0^\infty \frac{R_0^{\,2}}{r}\,\mathrm{d}r\right\}\right] \tag{4.23}$$

$$2\frac{\mathrm{d}\alpha_p}{\mathrm{d}t} = \frac{\gamma_p}{c^2}\left[[G_2]_0^\infty - p^2\left\{\int_0^\infty r\left(\frac{\mathrm{d}R_0}{\mathrm{d}r}\right)^2 \mathrm{d}r + \int_0^\infty \frac{R_0^{\,2}}{r}\,\mathrm{d}r\right\}\right] \tag{4.24}$$

where the brackets G_1 and G_2 are algebraically extensive. Including only terms which give a contribution when evaluated at the limits, after considerable cancellation, they are

$$G_1 = R_0^2\left(\frac{d}{dT}+\frac{1}{T}\right)M+\frac{R_0^2}{T}(Z_1+Z_3)+\frac{d\psi_1}{dT}\left(J+\frac{T}{2}\frac{dJ}{dT}-\tfrac{1}{2}\right)$$
$$+\psi_1\left(\frac{1}{2T}-\tfrac{1}{2}\frac{dJ}{dT}-\frac{J}{T}\right)-\frac{R_0^2}{2} \tag{4.25}$$

$$G_2 = R_0^2\left(\frac{d}{dT}+\frac{1}{T}\right)P-\frac{R_0^2}{T}(Z_1+Z_3)+\frac{d\psi_1}{dT}\left(L+\frac{T}{2}\frac{dL}{dT}+T\frac{dK}{dT}+\tfrac{1}{2}\right)$$
$$+\psi_1\left(\frac{1}{2T}-\tfrac{1}{2}\frac{dL}{dT}-\frac{L}{T}+\frac{dK}{dT}\right)+\frac{R_0^2}{2}. \tag{4.26}$$

Presumably one could deduce the results (4.23) and (4.24) from the basic equations for the inner solution, as was (3.16) in the rectilinear case. In view of the many complicated equations involved, however, this would be a difficult task.

For large T we find

$$G_1 = (p^2-1)\ln T+\left(2D_1+\frac{p^2}{2}+\tfrac{3}{4}\lambda+\tfrac{3}{16}\right)$$
$$G_2 = p^2\ln T+\left(2D_2+\frac{p^2}{2}-\tfrac{3}{4}\lambda-\tfrac{11}{16}\right).$$

Substitution into (4.23) and (4.24) then gives

$$-2\frac{d\gamma_p}{dt} = \frac{\alpha_p}{c^2}\left[\left(2D_1+\frac{p^2}{2}+\tfrac{3}{4}\lambda+\tfrac{3}{16}\right)-(p^2-1)\left\{\int_0^\infty r\left(\frac{dR_0}{dr}\right)^2 dr+⨍_0^\infty\frac{R_0^2}{r}dr\right\}\right] \tag{4.27a}$$

$$2\frac{d\alpha_p}{dt} = \frac{\gamma_p}{c^2}\left[\left(2D_2+\frac{p^2}{2}-\tfrac{3}{4}\lambda-\tfrac{11}{16}\right)-p^2\left\{\int_0^\infty r\left(\frac{dR_0}{dr}\right)^2 dr+⨍_0^\infty\frac{R_0^2}{r}dr\right\}\right] \tag{4.27b}$$

as G_1 and G_2 are both zero when evaluated at the lower limit. The constants D_1 and D_2 must be found by matching the interior and exterior solutions.

Exterior solution

Here we use the stretched coordinate $s = T/c$ for the outer variable so that as $c \to \infty$ with s of order unity we have the solution in the exterior region. In terms of this exterior variable the steady-state solution R_s, $\boldsymbol{u}_s$ is (cf. I § 3) for $s \to 0$

$$R_s = 1-\frac{1}{c^2}\left(\frac{1}{2s^2}+\frac{l}{2s}\cos\chi+\frac{\cos\chi}{s}(1-U_0c)+\ldots\right) \qquad \boldsymbol{u}_s = \frac{1}{c}(u_s, u_\chi, 0)$$

where

$$l = \ln\left(\frac{8}{s}\right)-2 \qquad u_s = \frac{l}{2}\sin\chi+\sin\chi(\tfrac{1}{2}-U_0c)+\left(\frac{3l}{8}+\tfrac{1}{4}\right)s\sin 2\chi$$

$$u_\chi = \frac{1}{s}+\frac{l}{2}\cos\chi+\cos\chi(1-U_0c)+\left(\frac{3l}{8}+\tfrac{7}{16}\right)s\cos 2\chi.$$

We now expand ϕ' and R' by writing

$$\phi' = \phi_{E1} + \frac{1}{c^2}\phi_{E2} + \dots \qquad R' = R_{E1} + \frac{1}{c^2}R_{E2} + \dots$$

and substitute into (2.3) and (2.4). Then equating the constant terms and coefficients of c^{-2} in (2.4) gives

$$R_{E1} = 0 \qquad R_{E2} = -\omega_2\phi_{E1} + \left(u_s\frac{\partial\phi_{E1}}{\partial s} + \frac{u_\chi}{s}\frac{\partial\phi_{E1}}{\partial\chi}\right) \tag{4.28}$$

while (2.3) also yields

$$\nabla^2\phi_{E1} = 0. \tag{4.29}$$

An appropriate solution of Laplace's equation (4.29) for a ring is given by Dyson (1893) as

$$\phi_p = \cos p\theta \int_0^{2\pi} \frac{\cos p\theta \, d\theta}{(z^2 + c^2 + \tilde{\omega}^2 - 2c\tilde{\omega}\cos\theta)^{1/2}}. \tag{4.30}$$

In addition $c(d\phi_p/dc)$ and $c(d\phi_p/dz)$ are also independent solutions of $\nabla^2\phi = 0$. J. J. Thomson, in his winning 1882 Adam's Prize essay, has given the value of the integral

$$b_p = \frac{1}{\pi}\int_0^{\pi} \frac{\cos p\theta \, d\theta}{(q - \cos\theta)^{1/2}}$$

when q is almost equal to unity (and $x \equiv q-1$ is therefore small) as

$$b_p = \frac{\sqrt{2}}{\pi}\left[F\left(\tfrac{1}{2}-p, \tfrac{1}{2}+p, 1, -\frac{x}{2}\right)\left(\ln\left(\frac{16(2+x)}{x}\right) - 4f(p)\right)\right.$$
$$\left. + \left\{K_1(p^2-\tfrac{1}{4})\frac{x}{2} + K_2(p^2-\tfrac{1}{4})(p^2-\tfrac{9}{4})\frac{x^2}{16} + \dots\right\}\right] \tag{4.31}$$

where $F(\frac{1}{2}-p, \frac{1}{2}+p, 1, -\frac{1}{2}x)$ is a hypergeometric series and

$$K_m = 2\sum_{n=1}^{m}\left(\frac{1}{n}\right).$$

From (4.30) it follows that

$$\phi_p = \frac{\pi}{2}\frac{\cos p\theta}{(2c\tilde{\omega})^{1/2}} b_p \tag{4.32}$$

and q is given by

$$q = (\tilde{\omega}^2 + c^2 - z^2)/2c\tilde{\omega}.$$

Thus

$$x = \frac{\tilde{\omega}^2 + c^2 - z^2 - 2c\tilde{\omega}}{2c\tilde{\omega}} = \frac{2T^2}{T_1^2 - T^2}$$

which is indeed small near the ring.

On substituting into (4.32) from (4.31) and using the expressions for $\ln(4T/T_1)$ and T^2/T_1^2 for small s from Dyson (p. 48), we obtain the expansion of ϕ_p as

$$\phi_p = \frac{\cos p\theta}{c}\left[l + 2(1-f(p)) + s\cos\chi\left(\frac{l}{2} + (\tfrac{1}{2} - f(p))\right) + s^2\left\{\frac{l}{4}(p^2+\tfrac{1}{2})\right.\right.$$
$$\left.\left. + \left(\frac{3p^2}{4} - p^2\frac{f(p)}{2} - \frac{f(p)}{4} + \tfrac{1}{16}\right)\right\} + s^2\cos 2\chi(\tfrac{3}{16}l + \tfrac{1}{8} - \tfrac{3}{8}f(p)) + \dots\right]. \tag{4.33}$$

The expansions of ϕ_p for small s for $p = 0$ and $p = 1$ have been given by Dyson (pp. 52–4) and for these values of p his expressions agree exactly with (4.33).

The solutions $c(\mathrm{d}\phi_p/\mathrm{d}c)$ and $c(\mathrm{d}\phi_p/\mathrm{d}z)$ can now be obtained by using the relations

$$\frac{\mathrm{d}}{\mathrm{d}c} = \frac{\partial}{\partial c} + \cos\chi\frac{\partial}{\partial T} - \frac{\sin\chi}{T}\frac{\partial}{\partial\chi} \qquad \frac{\mathrm{d}}{\mathrm{d}z} = \sin\chi\frac{\partial}{\partial T} + \frac{\cos\chi}{T}\frac{\partial}{\partial\chi}.$$

We eventually obtain

$$c\frac{\mathrm{d}\phi_p}{\mathrm{d}c} = -\frac{\cos p\theta}{c}\left[\frac{\cos\chi}{s} + \left\{\frac{l}{2} + (\tfrac{3}{4} - f(p))\right\} + \frac{\cos 2\chi}{4} + s\left\{-l\left(\frac{p^2}{2} - \tfrac{3}{8}\right) + f(p)(p^2 - \tfrac{3}{4}) - \frac{5p^2}{4} + \tfrac{11}{32}\right\}\cos\chi + \frac{3s}{32}\cos 3\chi + \mathrm{O}(ls^2)\right] \tag{4.34}$$

$$c\frac{\mathrm{d}\phi_p}{\mathrm{d}z} = -\frac{\cos p\theta}{c}\left[\frac{\sin\chi}{s} + \frac{\sin 2\chi}{4} + s\left\{-l\left(\frac{p^2}{2} - \tfrac{1}{8}\right) + f(p)(p^2 - \tfrac{1}{4}) - \frac{5p^2}{4} + \tfrac{5}{32}\right\}\sin\chi + \frac{3s}{32}\sin 3\chi + \mathrm{O}(ls^2)\right]. \tag{4.35}$$

Again expressions obtained by Dyson for $p = 0$ and $p = 1$ agree with these results. On multiplying (4.34) by $-\gamma_p$ and (4.35) by $-\alpha_p$ and adding we have the solution for ϕ_{E1} in the required form

$$\phi_{E1} = \cos p\theta\left(\frac{\alpha_p}{c}\left[\frac{\sin\chi}{s} + \frac{\sin 2\chi}{4} - ls\left(\frac{p^2}{2} - \tfrac{1}{8}\right)\sin\chi + s\sin\chi\left\{f(p)(p^2 - \tfrac{1}{4}) - \frac{5p^2}{4} + \tfrac{5}{32}\right\} + \frac{3s}{32}\sin 3\chi + \ldots\right] + \frac{\gamma_p}{c}\left[\frac{\cos\chi}{s} + \frac{l}{2} + (\tfrac{3}{4} - f(p)) + \frac{\cos 2\chi}{4} - ls\left(\frac{p^2}{2} - \tfrac{3}{8}\right)\cos\chi + s\cos\chi\left\{f(p)(p^2 - \tfrac{3}{4}) - \frac{5p^2}{4} + \tfrac{11}{32}\right\} + \frac{3s}{32}\cos 3\chi\ldots\right]\right). \tag{4.36}$$

Equation (4.28) then yields

$$R_{E2} = \cos p\theta\left(\frac{\alpha_p}{c}\left[\frac{\cos\chi}{s^3} + \frac{l}{2s^2}\cos 2\chi + \frac{1}{s^2}(\tfrac{5}{4} - U_0c)\cos 2\chi + \frac{1}{4s^2} - \frac{l}{s}\left(\frac{p^2}{2} - \tfrac{1}{4}\right)\cos\chi + \frac{l}{2s}\cos 3\chi + \frac{\cos\chi}{s}\left\{f(p)(p^2 - \tfrac{1}{4}) - \frac{5p^2}{4} - \frac{U_0c}{4} + \frac{\epsilon_2}{\epsilon_1}\omega_2 + \tfrac{1}{2}\right\} + \frac{\cos 3\chi}{s}\left(\tfrac{7}{8} - \frac{U_0c}{4}\right) + \ldots\right] - \frac{\gamma_p}{c}\left[\frac{\sin\chi}{s^3} + \frac{l}{2s^2}\sin 2\chi + \frac{1}{s^2}(\tfrac{5}{4} - U_0c)\sin 2\chi - \frac{l}{s}\left(\frac{p^2}{2} - \tfrac{3}{4}\right)\sin\chi + \frac{l}{2s}\sin 3\chi + \frac{\sin\chi}{s}\left\{f(p)(p^2 - \tfrac{3}{4}) - \frac{5p^2}{4} - \frac{3U_0c}{4} + \frac{\epsilon_1}{\epsilon_2}\omega_2 + \tfrac{3}{4}\right\} + \frac{\sin 3\chi}{s}\left(\tfrac{7}{8} - \frac{U_0c}{4}\right) + \ldots\right]\right). \tag{4.37}$$

Matching

In this case in order to express the inner solution for large T in terms of s we must replace T by sc and $\ln T$ by $\lambda - l$. Then from equations (4.16) to (4.19) and (4.22) we obtain the expansion of ϕ' as

$$\phi' = \cos p\theta\left(\frac{\alpha_p}{c}\left[\frac{\sin\chi}{s} + \frac{\sin 2\chi}{4} - ls\left(\frac{p^2}{2} - \tfrac{1}{8}\right)\sin\chi + s\left\{\left(\frac{p^2}{2} - \tfrac{1}{8}\right)\lambda + D_1\right\}\sin\chi \right.\right.$$
$$\left. + \frac{3s}{32}\sin 3\chi\ldots\right] + \frac{\gamma_p}{c}\left[\frac{\cos\chi}{s} + \frac{l}{2} + \left(\alpha - \frac{\lambda}{2}\right) + \frac{\cos 2\chi}{4} - ls\left(\frac{p^2}{2} - \tfrac{3}{8}\right)\cos\chi\right.$$
$$\left.\left. + s\left\{\left(\frac{p^2}{2} - \tfrac{3}{8}\right)\lambda + D_2\right\}\cos\chi + \frac{3s}{32}\cos 3\chi + \ldots\right] + O\left(\frac{1}{c^3}\right)\right). \tag{4.38}$$

A comparison of corresponding terms in (4.36) and (4.38) yields

$$D_1 = (p^2 - \tfrac{1}{4})\left(-\frac{\lambda}{2} + f(p)\right) - \frac{5p^2}{4} + \tfrac{5}{32} \qquad D_2 = (p^2 - \tfrac{3}{4})\left(-\frac{\lambda}{2} + f(p)\right) - \frac{5p^2}{4} + \tfrac{11}{32}. \tag{4.39}$$

We confirm (4.39) by matching the interior and exterior solutions for R' and in addition we obtain

$$U_0 = \frac{1}{2c}(\lambda - 2B + 1) \qquad 2F = \frac{3\lambda}{8} - \frac{B}{2} + \tfrac{1}{4}$$

which is consistent with previous work.

We are now able to substitute for D_1 and D_2 into (4.27). We find that

$$2\frac{d\gamma_p}{dt} = \frac{\alpha_p}{c^2}L_1 \tag{4.40a}$$

$$2\frac{d\alpha_p}{dt} = -\frac{\gamma_p}{c^2}L_2 \tag{4.40b}$$

where

$$L_1 = (p^2 - 1)(\ln(8c) - 2f(p) - \tfrac{1}{2} + \epsilon_c) - \tfrac{3}{2}(f(p) - 1)$$
$$L_2 = p^2(\ln(8c) - 2f(p) - \tfrac{1}{2} + \epsilon_c) + \tfrac{3}{2}f(p).$$

Differentiating (4.40b) and substituting for $d\gamma_p/dt$ from (4.40a) we finally obtain

$$\frac{d^2\alpha_p}{dt^2} = -\frac{\alpha_p}{4c^4}L_1L_2 \tag{4.41}$$

and the solutions of (4.41) and (4.40) are

$$\alpha_p = \epsilon_1\cos(\omega t + \mu) \qquad \gamma_p = \epsilon_2\sin(\omega t + \mu) \tag{4.42}$$

where

$$\omega = \frac{1}{2c^2}\{p^2(\ln(8c) - 2f(p) - \tfrac{1}{2} + \epsilon_c) + \tfrac{3}{2}f(p)\}^{1/2}\{(p^2 - 1)(\ln(8c) - 2f(p) - \tfrac{1}{2} + \epsilon_c)$$
$$- \tfrac{3}{2}(f(p) - 1)\}^{1/2} \tag{4.43}$$

$$\epsilon_2 = \epsilon_1\left(\frac{L_1}{L_2}\right)^{1/2}. \tag{4.44}$$

Thus the large vortex ring is stable for all small displacements of its circular axis, of the form (4.13). In terms of our physical variables our expression (4.43) for ω becomes (1.9) and the solutions may be compared with those obtained by J. J. Thomson (cf. equation (1.10) and equations (37), (42)) in his kinematical treatment of the large classical hollow core vortex ring. The expressions for ω differ only by the appearance of the ϵ_c term which again represents the effect of the core structure in the quantum vortex case

For the special cases $p = 0$ and $p = 1$ the results of our analysis contained in equations (4.40) have a particular physical interpretation. To see this we note that in the case $p = 0$ the initial vortex axis configuration (4.13) is $R_c = c+\alpha_p$, $Z_c = \gamma_p$ and this represents a small upward translation in the direction Oz and a slight increase in the radius of the ring. Altering the radius, however, automatically changes the speed of the vortex U_0 which is a function of c, to $U_0(c+\alpha_p)$. Then, since we are working in a frame in which the fluid at infinity is moving with velocity $U_0(c)$, the vortex will not remain at rest but will move with a speed $\alpha_p U_0'(c)$ relative to our fixed axes and consequently the distance of the vortex from the plane $z = 0$ increases linearly with time. This subsequent motion is confirmed by equations (4.40) which show that, for $p = 0$, $L_2 = 0$ so that by (4.40*b*)

$$\frac{d\alpha_p}{dt} = 0 \qquad \alpha_p = \delta_1 = \text{constant}.$$

Then by (4.40*a*) we have

$$\frac{d\gamma_p}{dt} = \delta_2 = \text{constant}$$

$$\gamma_p = \delta_2 t + \delta_3$$

just as we anticipate. Similarly the displacement given by $p = 1$ in (4.13) is simply a translation of the vortex in its own plane along $\theta = 0$ and a small tilt of the vortex about $\theta = 0$. This tilt slightly alters the direction of the vortex and in this case α_p increases linearly with time, a result which is again predicted by (4.40).

For large values of p we can neglect the terms in L_1 and L_2 of the form $Af(p)+B$ and the subsequent equations of the circular axis become

$$R_c = c+\epsilon_1 \cos p\theta \cos(\omega t+\mu)$$

$$Z_c = \epsilon_1 \left(\frac{p^2-1}{p^2}\right)^{1/2} \cos p\theta \sin(\omega t+\mu)$$

where

$$\omega = \frac{1}{2c^2}\{p^2(p^2-1)\}^{1/2}(\ln(8c)-2f(p)-\tfrac{1}{2}+\epsilon_c)$$

and the solutions are similar to those found by Arms and Hama, using their localized-induction concept (which corresponds to an omission of long distance effects), for a large vortex ring. They obtained

$$R_c = c+\epsilon_1 \cos p\theta \cos\left(\frac{\{p^2(p^2-1)\}^{1/2}}{c^2}t\right)$$

$$Z_c = \epsilon_1 \left(\frac{p^2-1}{p^2}\right)^{1/2} \cos p\theta \sin\left(\frac{\{p^2(p^2-1)\}^{1/2}}{c^2}t\right)$$

where the logarithmic behaviour is implicit in the time variable t which has been scaled by $\tfrac{1}{2}\ln 1/\epsilon$ where ϵ is a cut-off length.

If in addition $c \gg p(= kc) \gg 1$ we have

$$2f(p) \sim \gamma + 2\ln(2) + \ln p$$

and our expression for ω reduces to the rectilinear result of § 3. This is to be expected as this case is precisely that of a large number of long waves on an 'infinite' vortex.

Acknowledgments

I would like to express my thanks to Professor P. H. Roberts for many helpful discussions throughout this work and in particular for those valuable suggestions indicated in the text. I am also grateful to the Science Research Council for the award of a studentship.

References

DYSON, F. W., 1893, *Phil. Trans. R. Soc.* A, **185**, 43–95.
FETTER, A. L., 1965, *Phys. Rev.*, **138**, 709–16.
GINZBURG, V. L., and PITAEVSKII, L. P., 1958, *Sov. Phys.-JETP*, **7**, 858–61.
GROSS, E. P., 1961, *Nuovo Cim.*, **20**, 454–77.
—— 1963, *J. math. Phys.*, **4**, 195–207.
PITAEVSKII, L. P., 1961, *Sov. Phys.—JETP*, **13**, 451–4.
ROBERTS, P. H., and GRANT, J., 1971, *J. Phys.* A: *Gen Phys.*, **4**, 55–72.
THOMSON, J. J., 1883, *On the Motion of Vortex Rings* (London: Macmillan).
THOMSON, W., 1880, *Phil. Mag.*, Ser. 5, **10**, 155–68.